

Making Decisions with Spatially and Temporally Uncertain Data

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Abstract—We consider a decision-making problem where the environment varies both in space and time. Such problems arise naturally when considering e.g., the navigation of an underwater robot amidst ocean currents or the navigation of an aerial vehicle in wind. To model such spatiotemporal variation, we extend the standard Markov Decision Process (MDP) to a new framework called the Time-Varying Markov Decision Process (TVMDP). The TVMDP has a time-varying state transition model and transforms the standard MDP that considers only *immediate* and *static* uncertainty descriptions of state transitions, to a framework that is able to adapt to future time-varying uncertainty over some horizon. We show how to solve a TVMDP via a redesign of the MDP value propagation mechanisms by incorporating the introduced dynamics along the temporal dimension. We validate our framework in a marine robotics navigation setting using real spatiotemporal ocean data and show that it outperforms prior efforts to explicitly accommodate time by including it in the state.

I. INTRODUCTION

Consider a scenario where an underwater vehicle navigates across an area of ocean over a period of a few weeks to reach a goal location. Underwater vehicles such as autonomous gliders currently in use can travel long distances but move at speeds comparable to or slower than, typical ocean currents [31, 25]. These currents are typically strong enough (see Fig. 1) to disturb the vehicle’s motion, causing significantly uncertain action outcomes. Decision theoretic planning methods cope with action uncertainty by stochastically modeling the action outcomes. However, the dynamic nature of the ocean currents implies that to be effective, any underlying model that describes the uncertainty associated with the vehicle’s actions must also vary with time.

A popular method for addressing action uncertainty is the Markov Decision Process (MDP), a decision-theoretic planning framework in which the uncertainty due to actions is modeled using a stochastic state transition function (a probability distribution over resultant states when executing a particular action in a present state). However, a limitation of this approach is that the state transition model is static, i.e., the uncertainty distribution is a “snapshot at a certain moment” [21, 19].

Fortunately, environmental dynamics such as those associated with ocean currents can be forecast (albeit imprecisely) over a future time horizon. We exploit the idea that the forecast dynamics are time-varying functions that can be used to stochastically predict the state transition model. In this paper, we describe how to incorporate time-varying stochasticity into an MDP-style decision-making model and show how the resulting framework leads to improved planning accuracy.

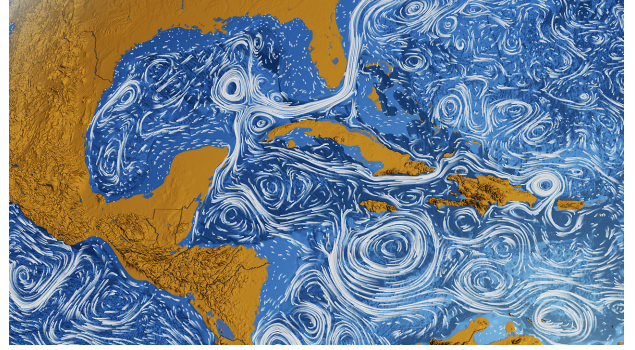


Fig. 1. Pattern of ocean currents (source: NASA). The ocean currents are time-varying and can be stochastically forecast over a future time horizon of a few days.

Specifically, this paper makes the following contributions:

- 1) We propose a new method called the *time-varying Markov Decision Process (TVMDP)*. The TVMDP has the capability of handling a time-varying transition model and can model decision-making problems with stochastic state transitions that vary both spatially and temporally, thus generalizing the standard MDP.
- 2) We show that in order to solve the TVMDP, the basic Bellman backup mechanism is not enough. Instead, we present a new mechanism based on two iterative value propagation processes, which proceed in both spatial and temporal dimensions.
- 3) From a robotics application perspective, we demonstrate how to characterize the underlying spatiotemporal dynamics from environmental data (here we use real ocean data), where the extracted ocean disturbance dynamics are incorporated into the TVMDP to describe the robot’s time-varying action uncertainty. The TVMDP is then solved to produce control policies for a robot navigating through the ocean data. Simulation results indicate that our solutions are more accurate, timely, and efficient compared to the classic MDP and a prior time-dependant variant.

II. RELATED WORK

Motion planning under uncertainty is an important functionality for autonomous robots operating in uncontrolled or unstructured environments. Planning under uncertainty problems can be formalized in a decision-theoretic framework [5, 10] where the two main sources of uncertainty arise from imperfect action outcomes and noisy sensors. Markov decision

processes (MDPs) and *partially observable* Markov decision processes (POMDPs) [8, 14, 22], have been extensively studied in decision-theoretic planning [19, 3].

Time varying Markov transition dynamics have been studied previously in the context of pattern analysis of economic growth [18, 2], (aimed at understanding the dynamics of growth based on a collection of different states corresponding to different countries), analyzing fiscal policies [1] (civilian spending/taxing), and the environment [13] (extreme temperature events). However, these models in these studies are Hidden Markov Models (HMMs), and unlike an MDP, they do not have the notion of an action to control transitions between states.

The approach proposed in this paper bears comparison to a framework called the *time-dependent Markov Decision Process* (TDMDP) [7] that includes time in the state space of the MDP. In the TDMDP the probability distribution associated with the transition model is used to describe the (uncertain) travel duration from one state to the next. This is similar to the conventional treatment of time-dependence in planning and routing in operations research [11]. In contrast, our approach (the TVMDP) follows the standard MDP convention and utilizes the transition model to describe the *uncertain transitions among states* (stochastic state jumping), **while assuming that the transition model itself is time-varying**. The TDMDP uses a likelihood function to control the state transitions. In order to take advantage of the standard Bellman backup mechanism to compute the solution, a set of restrictions are imposed on almost every term of the TDMDP formulation, making the framework less applicable in many realistic scenarios. Recently, the TDMDP has been analyzed [20], and it has been shown that even under strong assumptions and restrictions, the computation can still be cumbersome. The same analysis recommends means for approximating a solution using strategies such as prioritized sweeping [17]. The idea shares certain similarity to the *real-time dynamic programming* [4] where those states with larger values or value changes have a higher priority to propagate. We use this approximation scheme on TDMDP to compare with our approach in the experiment section. Differences between our approach and the TDMDP are compared more formally in the remainder of this work.

Proximal works also include the vast literature on reinforcement learning (RL) [28, 9, 15] wherein an agent tries to maximize accumulated rewards by interacting with the environment during its lifetime, and the environment is typically formulated as an MDP. A technique related to future prediction is temporal difference (TD) learning [27, 29]. RL extends the TD technique by allowing the learned state-values to guide actions, and correct previous predictions based on availability of new observations [28, 6]. Unfortunately, existing TD techniques are based on time-invariant models. Additionally, in order to extend the discrete-time actions, a temporal abstraction based concept (the Semi-Markov Decision Processes (SMDP)) [26] has been designed so that actions can be performed in a more flexible manner. SMDPs consider stochastic action durations, but still assume the transition model itself is time-invariant.

III. PRELIMINARIES

In this work, we assume that agents' states are fully observable and use a Markov Decision Process (MDP) to model and analyze the uncertainty and decision structures.

A. The Markov Decision Process (MDP)

Definition 1: An MDP \mathcal{M} is defined by a 4-tuple $\mathcal{M} = \langle S, A, T, R \rangle$, where $S = \{s\}$ and $A = \{a\}$ represent the countable state space and action space, respectively. The stochastic transition dynamics, also known as the transition model, are given by

$$T_a(s, s') = \Pr(s_{k+1} = s' | s_k = s, a_k = a) \quad (1)$$

which is a probability mass function that leads the agent to succeeding state $s_{k+1} = s'$ when it executes the action $a_k = a$ from state $s_k = s$. The time step k is also known as the computing epoch, where $1 \leq k \leq K$. The fourth element $R_a(s, s')$ is a positive scalar (also called the reward) for performing action a in state s and reaching state s' .

A control policy is a complete mapping $\pi : S \rightarrow A$ so that the agent applies the action $a_k \in A$ in state $s_k \in S$ at step k . If the action is independent of k , the policy is called *stationary* and a_k is simply denoted by a .

Note that, unlike conventional formulations, here we use k instead of t to index the algorithmic iterative steps/epochs, for both stationary and non-stationary models. In other words, we regard all K non-stationary algorithm iterations as momentary events, in order to distinguish the temporal process (a.k.a. planning horizon) which will be discussed further in the remainder of this paper.

Definition 2: the **Q-value** of a state-action pair (s, a) is defined as the one-step look-ahead value of state s if the immediate action a is performed. More formally,

$$Q(s, a) = \sum_{s' \in S} T_a(s, s') (R_a(s, s') + \gamma V(s')), \quad (2)$$

where $V(s')$ is the *value* (accumulated reward) for state s' and $\gamma \in [0, 1)$ is a discount factor for discounting future rewards at a geometric rate.

The objective is to find an optimal policy π^* satisfying

$$V_{\pi^*}(s) \equiv V^*(s) = \max_{a \in A} Q(s, a), \quad \forall s \in S. \quad (3)$$

When $\gamma < 1$, there exists a stationary policy that is optimal. V^* is the unique solution to the Bellman equations:

$$V^*(s) = \max_{a \in A} \sum_{s' \in S} T_a(s, s') (R_a(s, s') + \gamma V^*(s')). \quad (4)$$

and the optimal action policy $\pi^*(s)$ can be obtained

$$\pi^*(s) = \arg \max_{a \in A} \sum_{s' \in S} T_a(s, s') (R_a(s, s') + \gamma V^*(s')) \quad (5)$$

Employing Bellman's principle of optimality avoids enumerating solutions naively. In particular, *dynamic programming* based value iteration (VI) and policy iteration (PI) are the most widely used strategies for solving MDPs [19, 3].

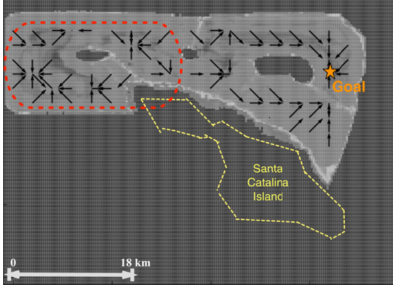


Fig. 2. A marine glider planning example. Policies (black arrows) are generated from a non-stationary MDP in which the future moments t are mapped to algorithmic epochs k . Closer inspection reveals that the policies could trap the robot in “local minima”.

B. The Transition Model

A conventional MDP has a static transition model $T_a(s, s')$. This means that T has no time-dependence (time is not an independent variable in the expression for T) nor does the form of T vary with time[†].

In many practical scenarios, e.g., marine vehicles in the ocean or aerial vehicles flying in the air, the transition model $T_a(s, s')$ must account for environmental disturbances that vary with time. Such a time variation property requires the control policy to also be a function of time to successfully reject dynamic disturbances and the resultant action uncertainty.

At first glance, it may appear that the standard MDP can be re-formulated by exploiting the non-stationary property. For instance, one may discretize the planning horizon into a series of time intervals. These time intervals can be sequentially associated to separate, successive, computing epochs so that such form of MDP can be solved using standard methods (e.g., value iteration or policy iteration). Fig. 2 demonstrates control policies computed using such a scheme. We can see that the policies form regions similar to the optimization local minima (some surrounding policies point inwards), which would trap the vehicle in arbitrary absorbing states. The reason is that the standard solution mechanisms are not able to interpret the temporal dynamics correctly. Therefore, modeling time-variation using a non-stationary MDP is not an effective way to proceed.

IV. TECHNICAL APPROACH

The analysis and concerns in the previous section motivate us to design a decision-making framework that involves dynamic state transition models.

A. Augment from “Spatial Only” to Both Spatial & Temporal

By “augmentation” we do not mean to simply append a time variable t to the state variable $s := \langle s, t \rangle$ so that the classic MDP framework can be directly employed. Such a simple appending operation is problematic:

[†] We use *time-dependent* and *time-varying* to mean different things. Following the convention in operations research, in the MDP context the term *time-dependent* is used to describe stochastic travel duration between states, whereas the term *time-varying* is used to express the change in the transition model as time elapses.

- 1) Time differs from space in its properties. Specifically, space is isotropic and symmetric. Within kinodynamic constraints, we can control an object to freely move in space. In contrast, time is asymmetric and unidirectional.
- 2) The differing properties of space and time imply that, *reachability* in space and time are not the same. We cannot travel back in time. We have much more restricted reachability (controllability) in time than in space.

The above comparison also explains why the standard solution for an MDP cannot be used for an MDP with a time-varying transition model. The standard MDP has no time constraints for state transitions. More importantly, the value propagation mechanisms, such as value iteration, assume that the value can be propagated in an arbitrary (even against the direction of time) direction. Therefore, appending a time variable to the state representation and combining them as one entity is inappropriate.

Correlate Space and Time:

The difference in reachability in space and time indicates that, the state transitions (and value propagation) in space and in time should be treated as two separate processes along a “spatial channel” and “temporal channel”, respectively.

The key idea of our work is to *evolve* the spatial process of state transition (and value propagation) along the temporal channel, where the spatial and temporal processes under two channels are coupled by the underlying real-world physics. For example, in the marine vehicle planning scenario, the two processes are correlated and coupled by the vehicle’s motion dynamics under environmental constraints such as disturbances from ocean currents.

Formally, to transform an MDP to a TVMDP, the static state transition model $T_a(s, s')$ needs to be a function of time, and we re-write it as $T_a(s, s', t)$. Similarly, the reward function becomes $R_a(s, s', t)$. The value function is modified accordingly:

$$V^*(s, t) = \max_{a \in A} \sum_{s' \in S} T_a(s, s', t) (R_a(s, s', t) + \gamma V^*(s', t')), \quad (6)$$

where $V(s', t')$ means the value of a future state s' at future time t' . Comparing Eq. (6) with the classic MDP (5), we see that every term of Eq. (6) is a function of time. Thus, the TVMDP can be written as $\mathcal{M}(t) = \langle S, A, T(t), R(t) \rangle$. Next, we are showing how to construct this time-varying decision-making model.

Add Real-Valued Transition Time:

A major difference between the TVMDP $\mathcal{M}(t)$ and the MDP \mathcal{M} lies in that, $\mathcal{M}(t)$ is built on, and thus requires estimation of, the *transition time* between pairwise states, which is different from the “state hopping” in an MDP – an important property inherited from Markov Chains.

The *transition time* here denotes the observable travel time of a robot from a state s to another state s' . It is used to track and map future moments to future policy rewards/values, along the unidirectional temporal channel.

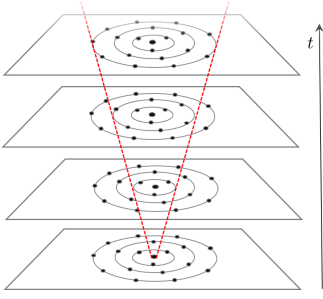


Fig. 3. Value iteration propagates along both spatial and temporal dimensions (red dashed lines), such that the future transition dynamics can be incorporated. Note that at different time layers the red lines are tangent to different ellipses, indicating that the spatial and temporal processes need to be coupled by robot's real world dynamics.

To understand the basic idea, assume that at time t_0 the robot is in state s_0 , and let $t(s_0, s)$ be the transition time from s_0 to an arbitrary state s . Since the transition model is a function of time, when the robot eventually arrives at state s , the transition dynamics that impact the robot are given by $T_a(s, s', \tau)$ where $\tau = t_0 + t(s_0, s)$ instead of $T_a(s_0, s)$ (or equivalently $T_a(s_0, s, t_0)$). Therefore, value evaluation/iteration for state s at the starting moment t_0 needs to utilize the transition model $T_a(s, s', \tau)$ captured at τ . Fig. 3 depicts such an idea. (It is worth mentioning that, the discrete time layers in Fig. 3 are used only for the purpose of demonstrating the idea of additional time dimension. We are showing that the proposed method does not need to discretize time.)

Also note that, here we use a perfect prediction model that assumes the estimated times $t(s_0, s)$ and τ are accurate. This however, is unrealistic due to the robot's stochastic behavior. This problem is addressed in Sect. IV-B.

Additionally, since the estimation of any real-valued transition time is based on continuous state space, we need to convert the conventional discrete MDP state space to a continuous one. There are many methods to do so [30, 23]. In this work, we adopt a simple strategy that maps from/to the other within certain resolution. The continuous form of action and external disturbance are also defined accordingly:

- State \mathbf{x} is the counterpart of s but in continuous state space. When \mathbf{x} coincides exactly at s , we denote it as $\mathbf{x}(s)$ in continuous space. We can also map \mathbf{x} back to discrete space: $\mathbf{x} \mapsto s$ if $\|\mathbf{x} - \mathbf{x}(s)\|$ is less than discrete space partition resolution;
- Local control/action reference $\mathbf{a}(s)$ at s is a directional vector, where the vector direction is defined by the discrete action $a \in A$, and the vector magnitude is defined by the robot's actual control inputs.
- Vector $\mathbf{d}(s)$ expresses external disturbance at s .

Consequently, if the transition from state s to a succeeding state s' is deterministic, the transition time can be approximated by $\|\mathbf{x}(s) - \mathbf{x}(s')\|/v$, or simply $\|\mathbf{x} - \mathbf{x}'\|/v$, assuming the robot's speed v is a constant in a static environment.

Construct Time-Varying Transition Models:

The mapping of future times to future policy values essentially relies on employing the correct transition models at those future moments. This is because policy values are propagated on the stochastic network that is expressed by those transition models. Hence, the key component of a TVMDP is the time-varying transition model. By a time-varying transition model, we mean that the state transition distribution is time-varying (e.g., it can be caused by the dynamic disturbances of the environment). This differentiates our proposed framework from the time-dependent formulations [7], in which the transition model is used to represent the stochastic travel duration between pairwise states.

Time-varying transition model can be predicted by utilizing the forecast environmental dynamics. For instance, the forecast data of ocean currents in next few days can define a predictable planning horizon, within which the extracted dynamics can be used to predict the vehicle's future state transitions.

For example, in many robotic motion planning scenarios, oftentimes the robot's action $\mathbf{a}(s)$ and external disturbance $\mathbf{d}(s)$ at state s may be additive (e.g., forces, velocities) and produce a resultant/net vector $\mathbf{r}(s) = \mathbf{a}(s) + \mathbf{d}(s)$ applied to the robot. Since both the robot's action and the forecast data can be imperfect, we assume $\mathbf{a}(s)$ and $\mathbf{d}(s)$ contain noise subject to independent Gaussian distributions. After a travel time \mathcal{T} , the robot's movement translation \mathbf{x} (denoting arriving state) after applying $\mathbf{a}(s)$ and being disturbed by $\mathbf{d}(s)$ also follows a Gaussian distribution:

$$f_a(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^d |\Sigma|}} \exp \left(-\frac{1}{2} (\mathbf{x} - \mu)^T \Sigma^{-1} (\mathbf{x} - \mu) \right) \quad (7)$$

where μ and Σ are the mean and covariance of \mathbf{x} , respectively. It is worth mentioning that the MDP may produce multiple optimal actions (with equal optimal value) at some state. In such a case, a *mixture distribution* can be used,

$$f_{\{a_1, \dots, a_k\}}(\mathbf{x}) = \sum_{i=1}^k w_i f_{a_i}(\mathbf{x}) = \frac{1}{k} \sum_{i=1}^k f_{a_i}(\mathbf{x}), \quad (8)$$

where the weighting parameter w_i for component PDFs are identical as actions have the same optimal value.

Let $\{\mathbf{a}^*(s)\}$ be the set of optimal actions at state s and at time t , i.e., s_t for short, the time-varying transition model thus can be expressed as

$$T_a(s, s', t) = \Pr(s' | s_t, \{\mathbf{a}^*(s_t)\}, \mathbf{d}(s_t)). \quad (9)$$

In practice, such discrete probability mass function is approximated by integrating Eq. (7) over discretized volumes.

B. Estimation of Transition Time

In order to estimate the time-varying transition models at different states at different future moments, the real-valued transition times from current state to all other states need to be estimated. Intuitively, the transition time estimation is a process of "look ahead into future", based on which the transition dynamics can then be predicted. One might also

imagine this as a *forward* process, as time evolves in a forward direction.

However, such a forward estimation cannot be done in a straightforward way. This is because the forward process/propagation is usually used on a deterministic acyclic graph, but in the MDP state transition topology, there are both cycles and stochasticity. The estimation from one state to another actually relies on each other, resulting in a chicken and egg problem.

In addition, we need to estimate not only the time of one-step look-ahead, but also the real-valued times corresponding to other distant states within the planning horizon.

Our approach consists of two steps. The first step is to estimate “local” transition time from each state $s \in S$ to its *one-hop* succeeding states $\mathcal{N}(s) \subseteq S$ that s can directly transit to. Building on this, the second step is to estimate the “global” transition time from the robot’s current state s_0 to all other *multi-hop* states $S \setminus \mathcal{N}(s_0)$.

Local One-Hop Transition Time Estimate:

The concept of one-hop can also be interpreted as one-step look ahead, since it measures the travel time to the adjacent neighboring states defined in the discrete space. The one-hop time estimate is *domain-specific* and may be done in multiple ways. For example, with a known transition model that essentially describes the state hopping probability distribution from the current state to neighboring states, the one-hop time can be tested offline by Monte Carlo trials, which can be cumbersome given many scenarios. A more efficient way is to obtain a closed form solution, where the time can be computed analytically based on the probability distribution information. In this work, we conduct the one-hop time estimate in the context of robotic motion planning, and our closed form analysis is as follows.

Due to the stochastic nature of the transition model, the robot in state s may eventually arrive at any one-hop succeeding state $s' \in \mathcal{N}(s)$. However, the estimation of transition time $t(s, s')$ is based on the assumption that the robot will reach a designated next state s' with probability 1. To satisfy this assumption, we need to choose an action $\tilde{\mathbf{a}}(s)$ with which the robot motion is exactly toward s' . Let $\tilde{\mathbf{r}}(s) = \tilde{\mathbf{a}}(s) + \mathbf{d}(s)$ denote the resultant of such selected action and environmental disturbance at s .

To simplify the calculation, one way is to transform the coordinate system such that the robot’s motion direction $\frac{\tilde{\mathbf{r}}(s)}{\|\tilde{\mathbf{r}}(s)\|}$ is exactly on an arbitrary coordinate basis, built on which the multivariate PDF can be approximated by a univariate PDF by marginalization.

Fig. 4 shows an illustration with robot state s located at the coordinate origin. θ is the angle between $\tilde{\mathbf{r}}(s)$ and a basis x_1 . To transform the coordinate system such that $\tilde{\mathbf{r}}(s)$ is on x_1 , the coordinate needs to rotate with corresponding rotation matrix $R(\theta)$. More generally, with a rotation matrix R and replacing $\mathbf{x} = R\tilde{\mathbf{x}}$ in Eq. (7), we obtain a distribution in the

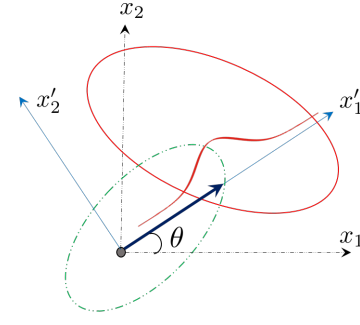


Fig. 4. Transition time estimation in 2D. The black dot at the origin represents the robot’s current state s ; the red solid ellipse denotes the state continuous distribution $f(\mathbf{x})$ under $\tilde{\mathbf{r}}(s) = \tilde{\mathbf{a}}(s) + \mathbf{d}(s)$ (the bold black arrow); the green dashed ellipse represents a contour of $\mathbf{r}(s) = \mathbf{a}(s) + \mathbf{d}(s)$ for all allowable $\mathbf{a}(s)$ given that $\mathbf{d}(s)$ is fixed.

transformed coordinate system:

$$\begin{aligned} f(\tilde{\mathbf{x}}) &\propto \exp\left(-\frac{1}{2}(\tilde{\mathbf{x}} - R^{-1}\mu)^T R^T \Sigma^{-1} R(\tilde{\mathbf{x}} - R^{-1}\mu)\right) \\ &\propto \exp\left(-\frac{1}{2}(\tilde{\mathbf{x}} - R^T \mu)^T (\tilde{\Sigma})^{-1} (\tilde{\mathbf{x}} - R^T \mu)\right) \end{aligned}$$

where $\tilde{\mu} = R^T \mu$ and $\tilde{\Sigma} = R^T \Sigma R$ after transformation.

Next we can calculate the *expectation* of resultant states in the robot’s motion direction. Let x_i denote a selected i -th basis (variable) in the direction of motion in the *new coordinate system*, and $\mathbf{x}_{-i} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_d)$ represent all other variables. Then the conditional expectation of x_i can be calculated by

$$\begin{aligned} \mathbf{E}(x_i | \mathbf{x}_{-i} = \mathbf{0}) &= \int x_i f(x_i | \mathbf{x}_{-i}) I_{\{\mathbf{x}_{-i} = \mathbf{0}\}}(\mathbf{x}_{-i}) d\mathbf{x}_{-i} \\ &= \int x_i \frac{f(x_i, \mathbf{x}_{-i})}{f(\mathbf{x}_{-i})} I_{\{\mathbf{x}_{-i} = \mathbf{0}\}}(\mathbf{x}_{-i}) d\mathbf{x}_{-i} \\ &= \int_0^\tau \int x_i \frac{f(x_i, \mathbf{x}_{-i})}{f(\mathbf{x}_{-i})} I_{\{\mathbf{x}_{-i} = \mathbf{0}\}}(\mathbf{x}_{-i}) d\mathbf{x}_{-i} dx_i \end{aligned}$$

where indicator variable $I_{\{\mathbf{x}_{-i} = \mathbf{0}\}}(\mathbf{x}_{-i}) = 1$ if $\mathbf{x}_{-i} = \mathbf{0}$ and 0 otherwise. τ can be either $+\infty$ (motion in the direction of x_i) or $-\infty$ (motion against the direction of x_i). Since states $\tilde{\mathbf{x}}$ are obtained by applying $\tilde{\mathbf{r}}(s)$ for a fixed time \mathcal{T} , the transition time $t(s, s')$ is approximated by

$$t(s, s') = \frac{\|\mathbf{x} - \mathbf{x}'\|}{(\mathbf{E}(x_i | \mathbf{x}_{-i} = \mathbf{0}) / \mathcal{T})} \quad (10)$$

where $\|\mathbf{x} - \mathbf{x}'\|$ represents the translation defined as before and the denominator is the estimated velocity given that the destination state is s' .

Global Multi-Hop Transition Time Estimate:

Using the one-hop transition time, we can estimate the global multi-hop transition time $t(s, s_e)$ from an arbitrary state s to an end state $s_e \in S$ that does not immediately succeed s . Multi-hop transition time estimation is *domain-independent*.

Estimating the global multi-hop transition time is challenging because the time estimate from an arbitrary state s to another multi-hop state s_e needs to take into account

many combinations of possible hopping scenarios due to the underlying cyclic nature. This causes interrelated dependence among states: estimation of arrival time at s_e travelled from s relies on a starting time at s , which essentially relies on the time estimates of all other states that directly or indirectly connect to s , including the state s_e .

One possible solution is to compute the *first passage* transition time at state s_e . However, the classic *Markov Chain first-passage times* method [24] does not apply here because it only considers and analyzes, the number/pattern of hops, instead of estimating the real-valued travel time.

We formulate the problem using Kolmogorov equations [16]. For example, from s_1 , the transition time $t(s_1, s_e)$ can be represented by an expectation $\mathbf{E}(t(s_1, s_j^{(1)}) + t(s_j^{(1)}, s_e))$, where $s_j^{(1)} \in \mathcal{N}(s_1)$ is s_1 's one-hop succeeding states, and $t(s_j^{(1)}, s_e)$ is again a multi-hop transition time from $s_j^{(1)}$ to ending state s_e . Similarly, we can formulate an expression for the transition time $t(s_i, s_e)$ for all other $s_i \in S$ to s_e :

$$\begin{cases} t(s_1, s_e) = \sum_{s_j^{(1)} \in \mathcal{N}(s_1)} T_a(s_1, s_j^{(1)}, t) (t(s_1, s_j^{(1)}) + t(s_j^{(1)}, s_e)) \\ \vdots \\ t(s_n, s_e) = \sum_{s_j^{(n)} \in \mathcal{N}(s_n)} T_a(s_n, s_j^{(n)}, t) (t(s_n, s_j^{(n)}) + t(s_j^{(n)}, s_e)) \end{cases}$$

where $t(s_e, s_e) = 0$ and each $t(s_i, s_j^{(i)})$ denotes the previously obtained one-hop transition time.

We have a total of $|S|$ variables and $|S|$ equations, which form a linear system. From the current state s_0 , there are $|S \setminus \mathcal{N}(s_0)|$ multi-hop non-succeeding states. Therefore, we need to solve $|S \setminus \mathcal{N}(s_0)|$ linear systems in each of which we specify a different end state $s_e \in S \setminus \mathcal{N}(s_0)$. Solving (sparse) linear systems is the most time expensive part during each value iteration, and it requires a complexity of $O(|S|^{2.3})$ [12]. Thus the overall time complexity to compute all $|S|$ estimates is $O(|S| * |S|^{2.3})$ for each iteration.

C. Synergy of Spatial and Temporal Processes

The final step to solve TVMDP is to propagate policy values in order to maximize optimal actions.

After having obtained the transition time t from the robot's current state s_0 to an arbitrary state s , the transition model at state s and time t can be constructed:

$$T_a(s, s', t) = T_a(s, s', t(s_0, s)). \quad (11)$$

where $s' \in S$ are the possible successor states from s .

In contrast to the forward time estimation process, the value propagation is more like a *backward* process: the propagation process employs a Bellman backup mechanism and propagates the rewards from distant states in the future back to the current state at the current moment. To improve estimation accuracy, the forward time estimate process and backward

value propagation process need to alternate iteratively until a certain (preset) convergence threshold is met.

After each iteration of Bellman backup, the action policy is updated. Based on the updated policy, the transition time estimates from the current state to all other states $t(s_0, s) \forall s \in S$ are calculated, which are then used to update the time-varying transition model $T_a(s, s', t(s_0, s))$ at s . The updated transition model is in turn utilized for next value iteration/propagation and time estimate. The proposed time estimation formulations produce unique and optimal solutions; and since value iteration is known to be near-optimal depending on the convergence stopping tolerance, the solution to TVMDP is thus near-optimal. However, the solution can become "suboptimal" when the time estimates have big errors compared to ground truth (e.g., forecast data and dynamics are inaccurate, vehicle state and control are noisy). The corresponding pseudo-code is described in Alg. 1.

In essence, the underlying computing mechanism can be imagined as value iteration that combines both spatial "expansion" and temporal "evolution", which occur simultaneously in two separate channels. This is a solution mechanism that fundamentally differs from that of the classic MDP.

Algorithm 1: Value Iteration (with agent's state s_0)

```

1 Initialize:  $k \leftarrow 1$ ,
2 foreach  $s \in S$  do
3    $\lfloor$  Initialize  $V_0(s) = 0, t(s_0, s) = 0$ 
4   /* value propagation in spatial channel */
5   foreach  $s \in S$  do
6      $\pi_k^*(s) = \operatorname{argmax}_{a \in A} \sum_{s' \in \mathcal{N}(s)} T_a(s, s', t(s_0, s)) \cdot$ 
7        $(R_a(s, s', t(s_0, s)) + \gamma V_{k-1}(s', t(s_0, s')))$ 
8     update optimal action  $a^*(s) = \pi_k^*(s)$ 
9   /* transition time estimates in temporal channel */
10  foreach  $s \in S \setminus \mathcal{N}(s_0)$  do
11    foreach  $s_j \in \mathcal{N}(s)$  do
12       $\lfloor$  Estimate one-hop transition time  $t(s, s_j)$ 
13     $\lfloor$  Estimate multi-hop transition time  $t(s_0, s)$ 
14   $k \leftarrow k + 1$ , goto Step 5.
15 Terminate if algorithm reaches given tolerance
```

V. EXPERIMENTS

We validated our method in an ocean monitoring scenario, where the ocean currents vary both spatially and temporally. An underwater glider simulator written in C++ was built in order to test the proposed decision-making framework. The simulation environment was constructed as a two dimensional ocean surface, and we tessellated the environment into a grid map. We represent the center of each cell/grid as a state, where each non-boundary state can transit in eight directions (N, NE, E, SE, S, SW, W, NW) and ninth idle action (returning to itself). Time-varying ocean currents are external disturbances

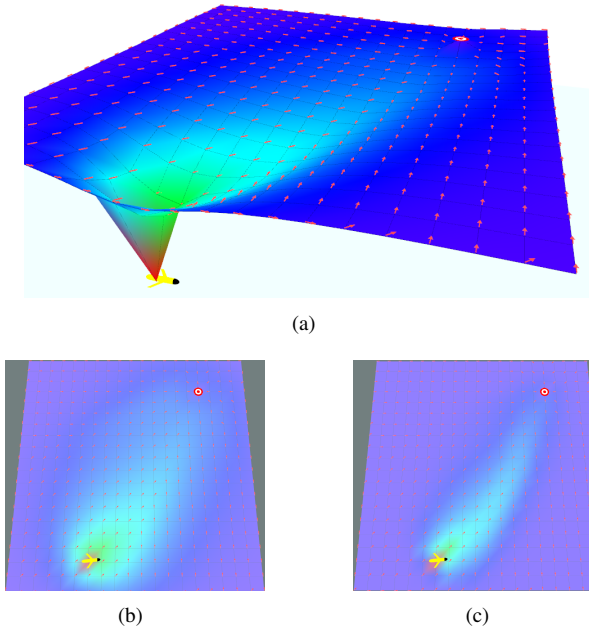


Fig. 5. (a) TVMDP action policy (red arrows) on a funnel-like surface where the color brightness represents estimated transition time from the robot’s state (funnel bottom); (b)(c) Policy maps projected onto 2D spatial dimensions. The colormap implies reachability in space. Different magnitudes of action uncertainty result in differing areas of reachable regions. The red target symbol is the goal state to reach.

for the robot and are represented as a vector field. Both the action and disturbance models are uncertain and are assumed to be Gaussian. Each resultant continuous state is mapped to a discrete tessellated state within resolution limits.

We first investigated the policy patterns generated from the proposed framework with time-varying transition models. Fig. 5(a) shows a policy map (red arrows) on a funnel-like surface where the color brightness is a measure of estimated transition time from the robot state (bottom of the funnel). Fig. 5(b) and 5(c) are projected policy maps onto a 2D plane. A brighter region implies a larger chance of being visited by the robot, and the difference of brighter regions in two figures reveals differing “magnitudes” of uncertainty (Fig. 5(b) has larger action uncertainty than Fig. 5(c)).

We then compared the methods with artificially created disturbances on a miniature grid map. Fig. 6(a) shows policies generated from the classic MDP where the underlying disturbance is static (not shown in the picture); In contrast, the policies in 6(b) are generated from the TVMDP, where the disturbance is constantly spinning. We can see that in TVMDP, not every action arrow is pointing to the goal. The interesting property of TVMDP is revealed in Fig. 6(c), which shows a predicted resultant vector field (i.e., $\mathbf{r}(s) = \mathbf{a}(s) + \mathbf{d}(s)$, $\forall s \in S$). We can observe a pattern similar to the policy map in Fig. 6(a) that is produced from the classic MDP under static disturbance. This indicates that, after a synergistic integration of spatial and temporal transition dynamics, the TVMDP “eliminates” the temporal dimensional dynamics and produces results similar to those of standard MDP with a static

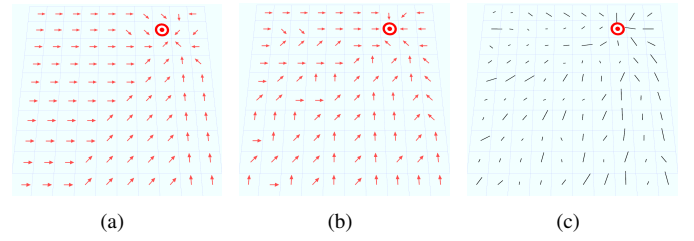


Fig. 6. (a) Policy map of the classic MDP under static disturbance; (b) Policy map of TVMDP under time-varying disturbance; (c) Predicted robot motion directions in TVMDP, after combining both robot action and time-varying disturbance. It reveals a pattern similar to the policy map of the classic MDP under static disturbance.

transition model.

Fig. 7 shows the robot’s trajectories resulting from different methods under different patterns of time-varying disturbances. Specifically, the left four figures are results from a spinning vector field of disturbance, whereas the right four ones are from vortex-like vector fields with translating vortex centers. We first compare between the standard MDP and TVMDP. Since MDP relies on a static transition model captured at a moment, we thus built the MDP’s transition model using the disturbance at the initial time frame. Fig 7(a)–7(d) demonstrate that the standard MDP performs badly in dynamic environments.

We then compared our approach with the time-dependent MDP (TMDP) framework [7]. Note that, in TMDP the transition model is used to describe stochastic durations and the likelihood that controls state transitions must be piecewise linear. These do not apply here since the environmental dynamics are usually in an arbitrary form. On top of the general model of TMDP, an approximation method using *prioritized sweeping* has been proposed to alleviate the prohibitive computational cost, while still assuming piecewise linear transition models [20]. To develop a control experiment, we borrow the idea of prioritized sweeping approximation but eliminate the piecewise linear restriction, and call this approach the *approximate time-varying MDP (ATMDP)*. The prioritized sweeping scheme essentially propagates the largest value function changes in priority through the state space, such that the dilemmas caused by the stochastic and cyclic topology are mitigated and a look-ahead time estimate solution is approximated. Fig. 7(e)–7(h) show trajectories produced from the ATMDP and TVMDP, respectively. We can see that, the trajectories of our method are much smoother and shorter.

Statistics with regard to trajectory lengths and time costs are provided in Fig. 8(a) and 8(b). These results indicate that the TVMDP method leads to smaller travel distances and shorter travel times. We use the Eigen iterative sparse matrix solver to compute linear systems for estimating multi-hop transition time. Fig. 8(c) shows that our method requires ~ 15 seconds for ~ 1000 states and ~ 50 seconds for ~ 2000 states (on a desktop with 3.30GHz Intel i7 CPU).

We also tested the algorithms with real ocean current data. The simulator is able to read and process data from the Re-

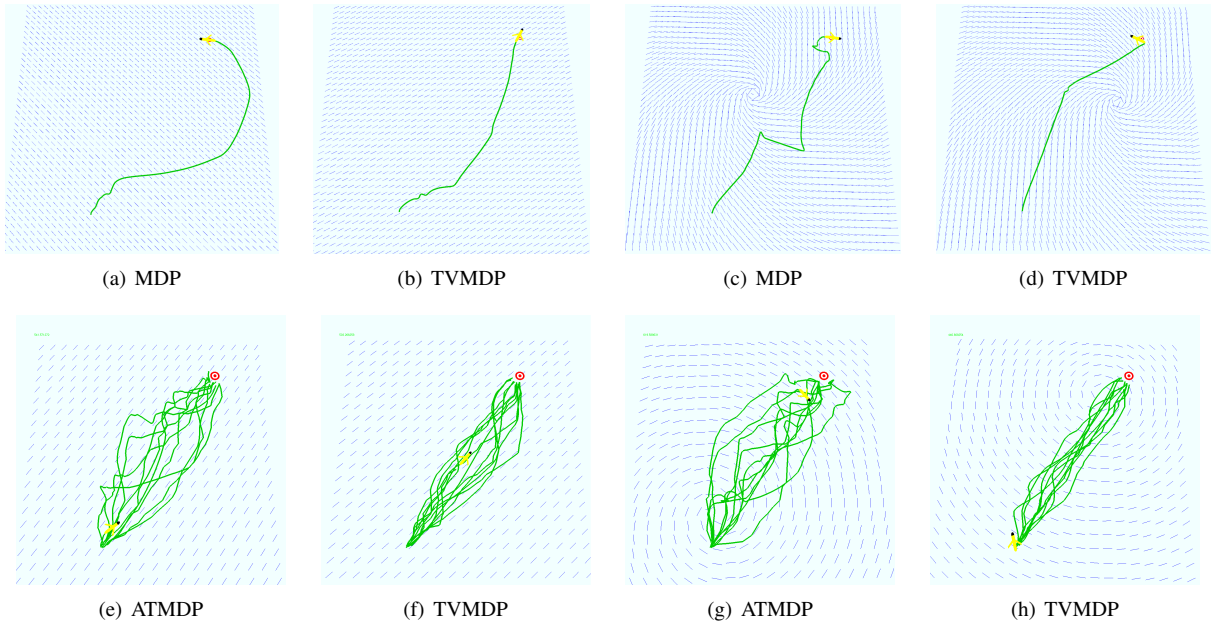


Fig. 7. (a)(b)(e)(f) Robot trajectories under a spinning disturbance vector field $\mathbf{d}([x, y]_t) = R(\omega t)[d, d]^T$, where $R(\omega t)$ is the rotation matrix with rotating rate ω and d is a constant value.; (c)(d)(g)(h) Trajectories under a vortex-like vector field $\mathbf{d}([x, y]_t) = [-(x - x_c(t)) + (y - y_c(t)), -(x - x_c(t)) - (y - y_c(t))]$ where $[x_c(t), y_c(t)]$ is a dynamic vortex center.

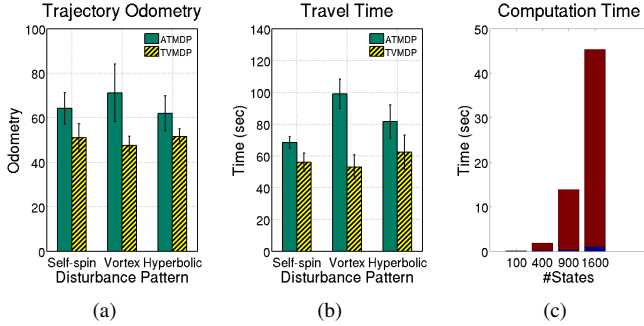


Fig. 8. Performance comparisons between ATMDP and the TVMDP, under differing disturbance patterns. (a) Trajectory odometry (lengths); (b) Overall travel time; (c) Computational time required by TVMDP to generate solutions. The red parts are the time used for solving linear systems.

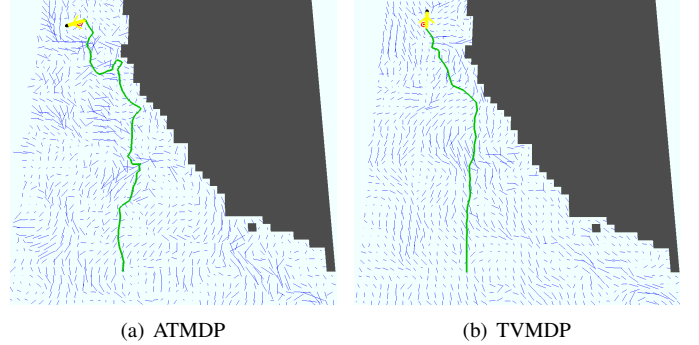


Fig. 9. Trajectory results from running on ROMS data.

VI. CONCLUSION

gional Ocean Model System (ROMS) which predicts/forecasts ocean currents up to 72 hours in advance. This allows us to utilize these ocean predictions to model the temporal dimensional transition dynamics. However, ROMS provides eight datasets for one day (every three hours). This means the data is not time-continuous. To address this, we use Gaussian Processes Regression (GPR) to interpolate and extrapolate the missing parts. Fig. 9 shows results from running the ATMDP and the TVMDP. We can observe that, comparing with TVMDP, only the beginning part of ATMDP trajectory is good. This is because ATMDP uses an approximation method that can only provide very rough time estimation results, and the approximation errors are propagated as the horizon grows, which eventually leads to poor control policies for the later part of the trajectory.

Markov Decision Processes as state-of-the-art decision-making methods are based on static and momentary stochastic transition dynamics, which lead to sub-optimal solutions if the transition dynamics vary with other parameters. We presented a time-varying MDP framework that takes into account a transition model that varies both spatially and temporally. We developed mechanisms to estimate the newly introduced temporal parameter as well as integrate the time-varying stochasticity. Finally we validated our method with various dynamic disturbances including those from real ocean data. The results show that the paths from our approach save both travel time and energy compared to the conventional method. It is worth mentioning that, although we built a model by incorporating temporal parameters, in general the model can be constructed analogously for any other new parameters and corresponding dimension augmentations.

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